# Optimal Precoding for Multiuser MIMO Systems With Phase Quantization and PSK Modulation via Branch-and-Bound

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***Abstract*—With an increasing number of antennas in MIMO systems, the energy consumption and costs of the corresponding front ends become relevant. In this context, a promising approach is the consideration of low-resolution data converters. In this study we propose an optimal precoding algorithm constrained to constant envelope signals and phase quantization that maximizes the minimum distance to the decision threshold at the receivers using a branch-and-bound strategy. The proposed algorithm is superior to the existing methods in terms of bit error rate. Numerical results show that the proposed approach has significantly lower complexity than exhaustive search.**

***Index Terms*—Precoding, low-resolution quantization, MIMO systems, branch-and-bound methods.**

## I. Introduction

One challenge in the wireless communications area is the minimization of the energy consumption and cost without major bit error rate performance compromise. In this context, systems with low-resolution quantizers are promising, as the energy consumption of data converters scales exponentially with the resolution in amplitude [1].

Several strategies for precoding with low-resolution quantizers exist. Linear approaches such as the Zero-forcing (ZF) [2] and Linear-MMSE method [3] have a relatively low complexity. Nevertheless, these methods often yield a detection error floor. Moreover, nonlinear precoders have been designed with different design criteria for achieving lower bit-error-rate (BER). A conventional design criterion is the MSE which is considered in the branch-and-bound (B&B) algorithm in [4]. Another widely used design criterion in given by the maximization of the minimum distance to the decision threshold (MMDDT) [5], [6], [7], [8], which is promising in combination with hard detection. In [7] an optimal precoding algorithm was presented for the MMDDT criterion and 1-bit quantization at transmitter and receiver (QPSK). In [8] a suboptimal algorithm is developed for the MMDDT criterion and $2^q$-PSK symbols at each transmit antenna for QAM and PSK modulation schemes.

In the present study, we generalize the work of [7] which uses 1-bit quantization, for phase quantizers with arbitrary number of phases at the transmit antennas and PSK modulation. This extension should be considered as non trivial because in the case of PSK, each symbol cannot be decomposed in independent real and imaginary part as done in the

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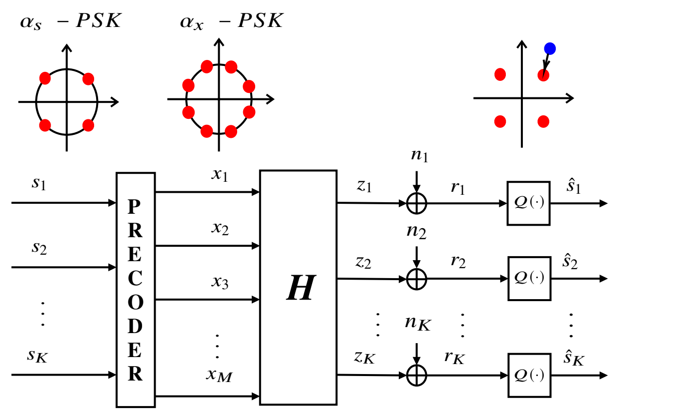


Fig. 1: Multiuser MIMO downlink with phase quantization and hard detection

1-bit case. The proposed precoder is optimal in terms of the MMDDT criterion, obtained by using a sophisticated branch-and-bound strategy.

In contrast to the method presented in [8], which implies a rounding step, the proposed algorithm finds the global optimum. Rounding steps when used for computing the final precoding vector causes BER performance degradation and can generate error floors. Considering that the global optimum has a maximized minimum distance to the decision threshold the proposed method approaches the minimum symbol-error-rate at high signal-to-noise ratio (SNR). Moreover, unlike the problem formulation in [8] with $2^q$ different phases, the proposed approach supports an arbitrary number of phase quantizations. In the initial step of the proposed method the relaxed problem is solved and rounded to the feasible set. Subsequently the optimum is determined by a tree search based algorithm.

The rest of the paper is organized as follows: Section II describes the system model, whereas Section III establishes the precoder's objectives, explains the criterion and exposes the problem formulation. In Section IV the proposed precoding algorithm is described. Section V presents and discusses numerical results, while Section VI gives the conclusions.

Regarding the notation, note that real and imaginary part operator are also applied to vectors and matrices, e.g., $\mathrm{Re}\{\boldsymbol{x}\} = [\mathrm{Re}\{[\boldsymbol{x}]_1\}, \ldots, \mathrm{Re}\{[\boldsymbol{x}]_M\}]^T$.

## II. System Model

In this study, a single cell MU-MIMO downlink with full channel state information at the base station (BS) is considered, as illustrated in Fig. 1. On the BS there are $M$

transmit antennas that serve $K$ single antenna users. The data symbol for the ith user $s_i$ is a $\alpha_s$-PSK symbol taken from the set $\mathcal{S}$ described by

$$\mathcal{S} = \left\{ s : s = e^{\frac{j\pi(2i+1)}{\alpha_s}}, \text{ for } i = 1, \ldots, \alpha_s \right\}. \tag{1}$$

The stacked vector with data symbols for the $K$ users is denoted by $\boldsymbol{s} = [s_1, \ldots, s_K]^T$. The vector $\boldsymbol{s}$ is the input for the precoder, where the transmit vector $\boldsymbol{x} = [x_1, \ldots, x_M]^T$ is constructed based on the channel. Due to the consideration of a low-resolution data converter at the transmitter, the entries from $\boldsymbol{x}$ are constrained to the set $\mathcal{X}$, which describes an $\alpha_x$-PSK alphabet given by

$$\mathcal{X} = \left\{ x : x = e^{\frac{j\pi(2i+1)}{\alpha_x}}, \text{ for } i = 1, \ldots, \alpha_x \right\}. \tag{2}$$

We consider analog pulse shaping filters at the BS and matched filtering, followed by a phase quantization process at the users. Moreover, we assume perfect synchronization. In the sequel the equivalent discrete time description of the channel is considered. A flat fading channel is considered, which is described by the matrix $\boldsymbol{H}$ whose coefficients $h_{k,m}$ are zero mean i.i.d. complex Gaussian random variables, where $k$ and $m$ denote the index of the user and the transmit antenna, respectively. With this, for the noiseless case the received signals are denoted by

$$z_k = \sum_{m=1}^{M} h_{k,m}\, x_m. \tag{3}$$

In the sequel a stacked vector notation is used with $\boldsymbol{z} = [z_1, \ldots, z_K]^T$.

At the receiver the signal $\boldsymbol{z}$ is corrupted by additive noise, which is denoted by the vector $\boldsymbol{n}$, which is considered to be a zero-mean i.i.d. complex Gaussian random vector with covariance matrix $\sigma_n^2 \boldsymbol{I}$.

With this, the vector of received samples can described by $\boldsymbol{r} = \boldsymbol{z}+\boldsymbol{n}$, which is subsequently processed by a phase quantizer that can be understood as a hard detector. In this regard, the received signal $\boldsymbol{r}$ is elementwise uniformly phase quantized. It is considered that number of quantization regions depends on the modulation alphabet of the data $\mathcal{S}$ with cardinality $\alpha_s$. The decision space is divided in $\alpha_s$ decision regions, where each possible symbol is associated with one region, as illustrated on the right hand side of Fig. 1. The decision regions are circle sectors with infinite radius and angle of $2\theta$, where $\theta$ is given by $\theta = \frac{\pi}{\alpha_s}$.

The output of the phase quantizer in stacked vector notation $\hat{\boldsymbol{s}} = [\hat{s}_1, \ldots, \hat{s}_K]^T$ is denoted by

$$\hat{\boldsymbol{s}} = Q(\boldsymbol{r}) = Q(\boldsymbol{z} + \boldsymbol{n}) = Q(\boldsymbol{Hx} + \boldsymbol{n}), \tag{4}$$

where $Q(\cdot)$ denotes the quantization operator. Each possible output represents an element of the transmit symbol alphabet ($\hat{\boldsymbol{s}} \in S^K$). With this, the vector $\boldsymbol{s}$ also represents the detected symbols $\hat{\boldsymbol{s}}$.

## III. Precoding objective

This section establishes the objective of the precoding algorithm and exposes the problem formulation. The criterion

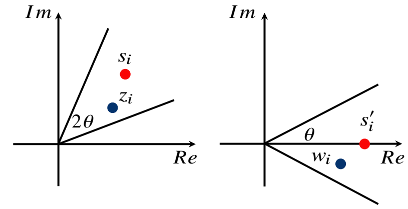


Fig. 2: Rotated coordinate system

for the precoder design is the maximization of the minimum distance to the decision threshold or equivalently the maximization of the safety margin at the detectors. With this, the aim is to find the vector $\boldsymbol{x}$ which yields the noiseless received vector $\boldsymbol{z}$, where the smallest distance to the decision threshold is maximized. The objective can be expressed in the epigraph form [9], which then corresponds to a linear objective function with linear constraints. Taking into account the quantization at each transmit antenna, the feasible set is discrete, which then yields a non-convex problem.

This study, relies on the maximization of the minimum distance to the decision threshold, denoted by $\epsilon$, for hard detection of PSK symbols and the description of the objective is equivalent to the one presented in [8]. Note that for the special case of QPSK modulation the objective is also equivalent to the objective utilized in [7].

By considering a rotation by $\arg\{s_i^*\} = -\phi_{s_i}$ of the coordinate system, the symbol of interest is placed on the real axis, as shown in Fig.2. This is done by multiplying both the interest symbol $s_i$ and the noiseless received signal $z_i$ by $e^{-j\phi_{s_i}} = s_i^*$ which reads

$$s_i^{'} = s_i s_i^* = 1, \quad w_i = z_i s_i^*. \tag{5}$$

The distance of the rotated symbol $w_i$ to the rotated decision threshold is then expressed as

$$\epsilon_i = \mathrm{Re}\left\{w_i\right\} \sin\theta - \left|\mathrm{Im}\left\{w_i\right\}\right| \cos\theta, \tag{6}$$

as shown in detail in [8]. Since the considered rotation includes also the decision thresholds the distance expression in (6) holds also for $z_i$. The minimum of all $\epsilon_i$, for $i = 1, \ldots, M$ is defined as $\epsilon$, which serves as the objective of the precoding design. The objective of the algorithm is to construct the transmit vector $\boldsymbol{x}$ that maximizes $\epsilon$. Based on a stacked vector notation for $w_i$, namely $\boldsymbol{w} = \mathrm{diag}(\boldsymbol{s}^*)\boldsymbol{Hx}$, the equivalent minimization problem reads

$$\begin{aligned} \left[\boldsymbol{x}_{\text{opt}}, \epsilon_{\text{opt}}\right] &= \underset{\boldsymbol{x} \in \mathcal{X}^M, \epsilon}{\arg\min} \; -\epsilon \\ &\text{s.t. } \; \mathrm{Re}\left\{\boldsymbol{H}_{s^*}\boldsymbol{x}\right\} \sin\theta - \left|\mathrm{Im}\left\{\boldsymbol{H}_{s^*}\boldsymbol{x}\right\}\right| \cos\theta \geq \epsilon \boldsymbol{1}_{2K}, \end{aligned} \tag{7}$$

where $\boldsymbol{H}_{s^*} = \mathrm{diag}(\boldsymbol{s}^*)\boldsymbol{H}$.

## IV. Proposed Branch-and-Bound Precoder

In this section we introduce the proposed precoder and derive the bounding steps for the algorithm. It is divided into three parts, the description of the mapped version of the MMDDT Precoder (MMDDT-Mapped), a general introduction of branch-and-bound precoding strategy and the description of the MMDDT branch-and-bound algorithm.

### A. MMDDT-Mapped Precoder

One approach for finding a feasible solution of (7) is to solve a relaxed version of the original problem followed by a mapping process to ensure that the precoding vector is in the feasible set of the discrete problem.

The relaxation is brought by replacing the set $\mathcal{X}^M$ by its convex hull, which then establishes convexity of the considered problem. The corresponding relaxed problem reads as

$$\begin{aligned} \left[\boldsymbol{x}_{\text{lb}},\, \epsilon_{\text{lb}}\right] &= \arg\min_{\boldsymbol{x},\epsilon} -\epsilon \\ &\text{s.t.}\quad \text{Re}\left\{\boldsymbol{H}_{s^*}\boldsymbol{x}\right\}\sin\theta - \left|\text{Im}\left\{\boldsymbol{H}_{s^*}\boldsymbol{x}\right\}\right|\cos\theta \geq \epsilon\boldsymbol{1}_{2K} \\ &\text{Re}\left\{x_m e^{j\phi_i}\right\} \leq \frac{\cos\left(\frac{\pi}{\alpha_x}\right)}{\sqrt{M}},\ \text{for } m=1,\ldots,M \text{ and} \\ &\phi_i = \frac{2\pi i}{\alpha_x},\ \text{for } i=1,\ldots\alpha_x, \end{aligned} \tag{8}$$

which is basically presented before in [8]. With the equivalent real valued notation, the problem in (8) can be expressed as a linear program (LP). Note that unlike the algorithm in [8], where $\alpha_x$ is restricted to integer powers of 2, the problem formulation (8) from above supports $\alpha_x$ to be any integer value. Subsequently the continuous solution $\boldsymbol{x}_{\text{lb}}$ is quantized to the point in $\mathcal{X}^M$ with the shortest Euclidean distance.

The optimal value of (8) is always a lower bound to the optimal value of the original problem (7). Mapping to the feasible set yields a valid solution $\boldsymbol{x}_{\text{ub}}$ and the corresponding value for $-\epsilon$ provides an upper bound on the optimal value of the original problem (7).

### B. Introduction of the Branch-and-Bound method

This part of the algorithm is a tree search problem, where a breadth first search is employed. For constructing the tree we consider that each node has $\alpha_x$ outgoing branches and that the tree consists of $M$ levels. The different levels in the tree correspond to the transmit antennas and the nodes in the $d$th level represent $d$ out of $M$ entries of a precoding vector in the discrete set.

For the computation of the discrete precoding vector we consider a constrained minimization of a precoding objective function $f(\boldsymbol{x},\boldsymbol{s})$, which could be the negative minimum distance to decision threshold, given by

$$\boldsymbol{x}_{\text{opt}} = \arg\min_{\boldsymbol{x}} f(\boldsymbol{x},\boldsymbol{s}) \quad \text{s.t. } \boldsymbol{x} \in \mathcal{X}^M. \tag{9}$$

A lower bound on $f(\boldsymbol{x}_{\text{opt}},\boldsymbol{s})$ can be obtained by relaxing this problem, e.g., as described in (8). An upper bound on $f(\boldsymbol{x}_{\text{opt}},\boldsymbol{s})$ can be found by mapping the solution of the relaxed version to $\mathcal{X}^M$ and evaluating $f(\cdot)$ accordingly. The upper bound on the optimal value is termed $\check{f}$.

Note that, since $\check{f} \geq f(\boldsymbol{x}_{\text{opt}},\boldsymbol{s}) \geq f(\boldsymbol{x}_{\text{lb}},\boldsymbol{s})$, the mapped solution, cannot yield a better solution than the relaxed solution.

If we consider $d$ fixed entries of $\boldsymbol{x}$, the precoding vector becomes $\boldsymbol{x} = [\boldsymbol{x}_1^T, \boldsymbol{x}_2^T]^T$, with $\boldsymbol{x}_1 \in \mathcal{X}^d$. Then a sub problem can be formulated with

$$\begin{aligned} \boldsymbol{x}_{2,\text{lb}} &= \arg\min_{\boldsymbol{x}_2} f(\boldsymbol{x}_2,\boldsymbol{x}_1,\boldsymbol{s}) \\ &\text{s.t. } \text{Re}\left\{x_m e^{j\phi_i}\right\} \leq \frac{\cos\left(\frac{\pi}{\alpha_x}\right)}{\sqrt{M}}, \text{for } m=1,\ldots,M-d \\ &\phi_i = \frac{2\pi i}{\alpha_x},\ \text{for } i=1,\ldots\alpha_x\ . \end{aligned} \tag{10}$$

If the optimal value of (10) is larger (worse) than a known upper bound $\check{f}$ on the solution of (9), then all member in the discrete solution set which include vector $\boldsymbol{x}_1$ can be excluded from the search. In the context of the proposed tree search, the vector $\boldsymbol{x}_1$ is defined by the different nodes in the tree.

### C. MMDDT Branch-and-Bound algorithm derivation

In this section a branch-and-bound algorithm is proposed which solves (9) by considering the problem in (8) for the initialization and sub problems as given by (10) for computing lower bounds.

In order to formulate a real valued problem matrix $\boldsymbol{H}_r$ and vector $\boldsymbol{x}_r$ are defined as follows

$$\boldsymbol{x}_{\text{r}} = \begin{bmatrix} \text{Re}\left\{\boldsymbol{x}_1\right\} \\ \text{Im}\left\{\boldsymbol{x}_1\right\} \\ \text{Re}\left\{\boldsymbol{x}_2\right\} \\ \text{Im}\left\{\boldsymbol{x}_2\right\} \\ \vdots \\ \text{Re}\left\{\boldsymbol{x}_M\right\} \\ \text{Im}\left\{\boldsymbol{x}_M\right\} \end{bmatrix}, \quad \boldsymbol{H}_{\text{r}} = \begin{bmatrix} \Gamma_{11}\cdots\Gamma_{1M} \\ \Lambda_{11}\cdots\Lambda_{1M} \\ \vdots \\ \Gamma_{K1}\cdots\Gamma_{KM} \\ \Lambda_{K1}\cdots\Lambda_{KM} \\ \Psi_{11}\cdots\ \Psi_{1M} \\ \Delta_{11}\cdots\Delta_{1M} \\ \vdots \\ \Psi_{K1}\cdots\Psi_{KM} \\ \Delta_{K1}\cdots\Delta_{KM} \end{bmatrix}, \tag{11}$$

with

$$\begin{aligned} \boldsymbol{\Gamma} &= \quad \text{Im}\left\{\boldsymbol{H}_{s^*}\right\}\cos(\theta) - \text{Re}\left\{\boldsymbol{H}_{s^*}\right\}\sin(\theta) \\ \boldsymbol{\Lambda} &= \quad \text{Re}\left\{\boldsymbol{H}_{s^*}\right\}\cos(\theta) + \text{Im}\left\{\boldsymbol{H}_{s^*}\right\}\sin(\theta) \\ \boldsymbol{\Psi} &= -\text{Im}\left\{\boldsymbol{H}_{s^*}\right\}\cos(\theta) - \text{Re}\left\{\boldsymbol{H}_{s^*}\right\}\sin(\theta) \\ \boldsymbol{\Delta} &= \quad \text{Im}\left\{\boldsymbol{H}_{s^*}\right\}\sin(\theta) - \text{Re}\left\{\boldsymbol{H}_{s^*}\right\}\cos(\theta). \end{aligned} \tag{12}$$

With the real valued notation, the variable vector of the optimization problem with length $2M+1$ can be denoted by $\boldsymbol{v} = [\epsilon, \boldsymbol{x}_{\text{r}}^T]^T$. With this, the real valued problem reads as

$$\begin{aligned} \boldsymbol{v}_{\text{opt}} &= \arg\min_{\boldsymbol{v}} \boldsymbol{a}^T\boldsymbol{v} \\ &\text{s.t. } \boldsymbol{A}\boldsymbol{v} \leq \boldsymbol{0}_{2K}, \\ &\left\{\boldsymbol{v}_{2m} + j\boldsymbol{v}_{2m+1}\right\} \in \mathcal{X}, \quad \text{for } m=1,\ldots,M, \end{aligned} \tag{13}$$

with

$$\boldsymbol{a} = [-1, \boldsymbol{0}_{2M}^T]^T, \quad \boldsymbol{A} = \left[\boldsymbol{1}_{2K}, \boldsymbol{H}_{\text{r}}\right].$$

Replacing the the discrete solution set by its convex hull yields the relaxed problem given by

$$\boldsymbol{v}_{\text{lb}} = \arg\min_{\boldsymbol{v}} \boldsymbol{a}^T\boldsymbol{v} \quad \text{s.t. } \boldsymbol{U}\boldsymbol{v} \leq \boldsymbol{p}, \tag{14}$$

with

$$\boldsymbol{U} = \left[\boldsymbol{A}^T, \boldsymbol{R}^T\right]^T \qquad \boldsymbol{R} = \left[\boldsymbol{0}_{M\,\alpha_x}, \boldsymbol{R}'\right]$$

$$\boldsymbol{R}' = \left[ (\boldsymbol{I}_M \otimes \boldsymbol{\beta}_1)^T, (\boldsymbol{I}_M \otimes \boldsymbol{\beta}_2)^T, \ldots, (\boldsymbol{I}_M \otimes \boldsymbol{\beta}_{\alpha_x})^T \right]^T$$

$$\boldsymbol{\beta}_i = \left[\cos\phi_i, -\sin\phi_i\right] \quad \boldsymbol{p} = \left[\boldsymbol{0}_{2K}, \frac{\cos(\frac{\pi}{\alpha_x})}{\sqrt{M}} \boldsymbol{1}_{M\alpha_x}\right]^T.$$

In the branch-and-bound method in each visited node sub problems are solved due to $\boldsymbol{v} = \left[\epsilon, \boldsymbol{x}_{r_1}^T, \boldsymbol{x}_{r_2}^T\right]^T$, where $\boldsymbol{x}_{r_1}$ is a fixed vector of length $2d$, which belongs to the discrete set according to $\boldsymbol{v}_{1_{2m}} + j\boldsymbol{v}_{1_{2m+1}} \in \mathcal{X}$, for $m = 1, \ldots, d$.

The matrix $\boldsymbol{U}$ can be expressed with the following structure $\boldsymbol{U} = [\boldsymbol{u}_1, \boldsymbol{U}_1, \boldsymbol{U}_2]$, where $\boldsymbol{U}_1$ contains $2d$ columns of $\boldsymbol{U}$ and $\boldsymbol{u}_1$ is the first column of $\boldsymbol{U}$. With this, the matrix $\tilde{\boldsymbol{U}} = \left[\boldsymbol{u}_1, \boldsymbol{U}_2\right]$ and the vector $\tilde{\boldsymbol{v}} = \left[\epsilon, \boldsymbol{x}_{r_2}^T\right]^T$ are composed. Using $\tilde{\boldsymbol{U}}$ and $\tilde{\boldsymbol{v}}$ the sub problem for the lower-bounding step can be expressed as

$$\tilde{\boldsymbol{v}}_{\text{lb}} = \arg\min_{\tilde{\boldsymbol{v}}} \tilde{\boldsymbol{a}}^T \tilde{\boldsymbol{v}} \quad \text{s.t. } \tilde{\boldsymbol{U}}\tilde{\boldsymbol{v}} \leq \boldsymbol{b}, \tag{15}$$

with $\tilde{\boldsymbol{a}} = \left[-1, \boldsymbol{0}_{2M-2d}^T\right]^T$ and $\boldsymbol{b} = \boldsymbol{p} - \boldsymbol{U}_1\boldsymbol{x}_{r_1}$. Solving (15) provides a lower bound on the optimal value of the discrete problem with the condition on $\boldsymbol{x}_{r_1}$. In case the lower bound conditioned on $\boldsymbol{x}_{r_1}$ is higher than any upper bound on the original problem $\boldsymbol{x}_{r_1}$ cannot be part of the solution and every member of the discrete solution set which includes $\boldsymbol{x}_{r_1}$ can be excluded from the search. In the context of the tree search it means that once a partial solution $\boldsymbol{x}_{r_1}$ is excluded the corresponding node and all its evolutions can be skipped. The steps of the method are detailed in Algorithm 1. Note that the computation of the optimal precoding vector in each symbol period can correspond to an enormous computational complexity. Nevertheless, the method might be a practical solution for channels with large coherence time, where the finite number of different precoding vectors can be precomputed and stored as suggested in [10].

**Algorithm 1** Proposed B&B Precoding for solving (7)

**initialization:**
Given the channel $\boldsymbol{H}$ and transmit symbols $\boldsymbol{s}$ compute a valid upper bound $\check{f}$ on the problem in (7), e.g., by solving (8) followed by a mapping to the closest precoding vector $\boldsymbol{x} \in \mathcal{X}^M$

Define the first level ($d = 1$) of the tree by $\mathcal{G}_d := \mathcal{X}$

**for** $d = 1 : M-1$ **do**
  Partition $\mathcal{G}_d$ in $\boldsymbol{x}_{1,1}, \ldots, \boldsymbol{x}_{1,|\mathcal{G}_d|}$
  **for** $i = 1 : |\mathcal{G}_d|$ **do**
    Express $\boldsymbol{x}_{1,i}$ with stacked vector notation due to (11) as $\boldsymbol{x}_{\boldsymbol{r}1,i}$
    Conditioned on $\boldsymbol{x}_{\boldsymbol{r}1,i}$ solve $\tilde{\boldsymbol{v}}_{\text{lb}}$ from (15)
    Determine $\epsilon = [\tilde{\boldsymbol{v}}_{\text{lb}}]_1$
    Compute the lower bound: $\text{lb}(\boldsymbol{x}_{1,i}) := -\epsilon$;
    Map $\boldsymbol{x}_{2,\text{lb}}$ to the discrete solution with the closest Euclidean distance:
    $\check{\boldsymbol{x}}_2(\boldsymbol{x}_{2,\text{lb}}) \in \mathcal{X}^{M-d}$
    Using $\check{\boldsymbol{x}}_2$ find the smallest (negative) distance to the decision threshold $\text{ub}(\boldsymbol{x}_{1,i}) :=$

$$\max_k \left[ \left| \text{Im}\left\{ \boldsymbol{H}_{s^*} \begin{bmatrix} \boldsymbol{x}_{1,i} \\ \check{\boldsymbol{x}}_2 \end{bmatrix} \right\} \right| \cos\theta \} - \text{Re}\left\{ \boldsymbol{H}_{s^*} \begin{bmatrix} \boldsymbol{x}_{1,i} \\ \check{\boldsymbol{x}}_2 \end{bmatrix} \right\} \sin\theta \right]_k$$

    Update the best upper bound with:
    $\check{f} = \min\left(\check{f}, \text{ub}(\boldsymbol{x}_{1,i})\right)$
  **end for**
  Build a reduced set by comparing conditioned lower bounds with the global upper bound $\check{f}$
  $\mathcal{G}'_d := \left\{\boldsymbol{x}_{2,i} | \text{lb}(\boldsymbol{x}_{2,i}) \leq \check{f}, i = 1, \ldots, |\mathcal{G}_d|\right\}$
  Define the set for the next level in the tree
  $\mathcal{G}_{d+1} := \mathcal{G}'_d \times \mathcal{X}$
**end for**
Search method for the ultimate level $d = M$,
Partition $\mathcal{G}_1$ in $\boldsymbol{x}_{1,1}, \ldots, \boldsymbol{x}_{1,|\mathcal{G}_1|}$

$$\epsilon(\boldsymbol{x}_{1,i}) := \min_k \left[\text{Re}\left\{\boldsymbol{H}_{s^*}\boldsymbol{x}_{1,i}\right\}\sin\theta - |\text{Im}\left\{\boldsymbol{H}_{s^*}\boldsymbol{x}_{1,i}\right\}|\cos\theta\right]_k$$

The global solution is

$$\boldsymbol{x}_{\text{opt}} = \underset{\boldsymbol{x}_{1,i} \in \mathcal{G}_1}{\text{argmax}} \ \epsilon(\boldsymbol{x}_{1,i})$$

## V. Numerical Results

For the numerical evaluation the uncoded bit error rate is considered, where Gray-coding is used. The considered SNR is defined by $\text{SNR} = \frac{\|\boldsymbol{x}\|_2^2}{N_0}$, where the spectral noise power density $N_0$ is equivalent to the noise sample variance $\sigma_n^2$, as our model implicitly takes into account a receive filter with unit energy normalization. For numerical computations $K = 3$ users and 1000 random channel realizations are considered.

The proposed method is compared with the following algorithms from the literature: 1. The MSM-Precoder [8] considering phase quantization which implies solving an LP; 2. The ZF precoder with constant envelope [2], where the entries of the precoding vector are subsequently phase quantized; 3. The phase quantized CIO precoder implemented via CVX [11], which corresponds to solving a second order cone program. The corresponding computational complexity is summarized in Table I, where $N$ denotes the number of evaluated bounds in the proposed algorithm. In addition, the MMDDT precoder with full resolution and per antenna power constrained is considered, which yields a higher optimal value for $\epsilon$, because relaxation of the feasible set results in an upper bound of the optimal value of the original problem.

In Fig. 3 a conventional configuration is considered with 8-PSK symbols ($\alpha_x = 8$, $\alpha_s = 8$). For different numbers of BS antennas ($M = 7$, $M = 14$) it is shown that the proposed algorithm has a significantly lower BER than existing suboptimal algorithms, which confirms the aptitude of the MMDDT design criterion in the context of hard detection. The proposed algorithm does not show an error floor, which occurs for the corresponding suboptimal precoding algorithm [8]. This indicates that the rounding step can correspond to a significant performance degradation.

In addition, to demonstrate the flexibility of the proposed framework, a more exotic configuration is considered in Fig. 4, where $x_i$ is a 3-PSK symbol using QPSK modulation at the same time ($\alpha_x = 3$, $\alpha_s = 4$), which is compatible only with a subset of the existing methods.

The proposed branch-and-bound method yields the same solution as the exhaustive search but with a lower average

TABLE I: Computational Complexity of the Algorithms

| Algorithm | Complexity |
|---|---|
| ZF-P [2] | $O(K^2M)$ |
| CVX-CIO [11] | $O((2M+1)^{3.5})$ |
| MSM-Precoder [8] | $O((2M+1)^{3.5})$ |
| Proposed MMDDT B&B | $O(N\,(2M+1)^{3.5})$ |

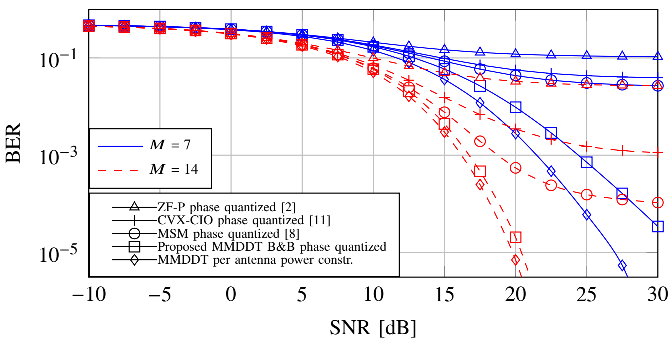


Fig. 3: Uncoded BER versus SNR, $K = 3$, $\alpha_s = 8$ and $\alpha_x = 8$

complexity. The complexity of the algorithm depends on finding as early as possible a tight upper bound that permits many exclusions of possible candidates while going down the tree. Using interior point methods (IPM) for solving sub problems (15) corresponds to a computational complexity of $O(l^{3.5})$, with $l \leq (2M+1)$. Note that the dimensions of the sub problems decrease when climbing down the tree.

The computational complexity in terms of the number of solved subproblems $N$ is shown in Fig. 5. Fig. 5a illustrates that the average number of solved sub problems ($\overline{N}$) is only a small fraction of the number of candidates which are evaluated in the exhaustive search. Taking into account that each candidate evaluation in the exhaustive search corresponds to a complexity of $O(MK)$, justifies the utilization of the proposed branch-and-bound approach for determining the optimal precoding vector. Moreover, $\overline{N}$ is illustrated in Fig. 5b and its standard deviation ($\sigma_N$) is shown in Fig. 5c, which indicates that the complexity scales less than exponentially with $M$.

## VI. Conclusions

An optimal algorithm for precoding constrained to constant envelope and phase quantization for PSK modulation and hard detection is proposed. The design criterion maximizes the minimum distance to the decision threshold at the receivers. The proposed algorithm outperforms the state-of-art techniques for this class of precoding in terms of BER. Numerical results confirm the efficiency of the proposed branch-and-bound strategy.

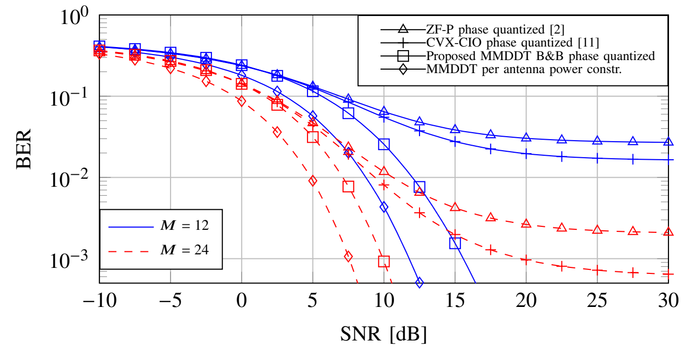


Fig. 4: Uncoded BER versus SNR, $K = 3$, $\alpha_s = 4$ and $\alpha_x = 3$

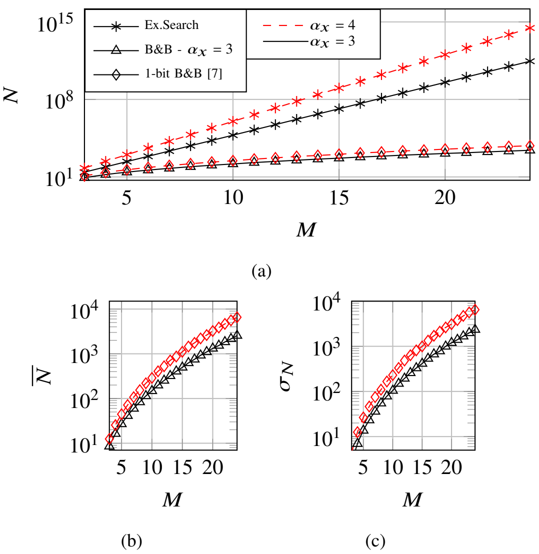


Fig. 5: Complexity in terms # of evaluated bounds $N$, $K = 3$ users, $\alpha_s = 4$, (a) Average of $N$ in comparison to exhaustive search, (b) Behaviour of the average of $N$, (c) Standard deviation of $N$

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